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ON CERTAIN PROPERTIES OF THE EQUATIONS OF THE THREE-DIMENSIONAL BOUNDARY LAYER

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The paper deals with a system of equations of a three-dimensional boundary layer (see e. g. [1]) with the conditions of suction or adhesion prevailing at the surface of the body. It is shown that for the negative values of the longitudinal and transverse components of the pressure gradient and the corresponding boundary conditions, the longitudinal and transverse components of the velocity vector are positive not only near the body, but also throughout the thickness of the boundary layer. This makes possible the estimation of the upper limit of the velocity components in question in relation to the boundary conditions, and to establish the behavior of the projections of the streamlines on the surface of the body. The latter is of interest in connection with the domains of influence and domains of dependence in the three-dimensional boundary layer [2–5].

Let $G \subset R^4(x, y, z, t)$ be a region defined by the conditions $0 < x < X$, $0 < y < Y$, $0 < z < Z$, $0 < t < T$. We consider in G a system of equations with the boundary conditions

$$\nu u_{yy} - uu_x - vu_y - wu_z - u_t = p_x/\rho \tag{1}$$

$$\nu w_{yy} - uw_x - vw_y - ww_z - w_t = p_z/\rho$$

$$u_x + v_y + w_z = 0$$

$$u = w = 0, \quad v = V(x, z, t) \leqslant 0, \quad y = 0$$

$$u = U, \ w = W \text{ on } \partial G \cap [\{x = 0\} \cup \{z = 0\} \cup \{t = 0\} \cup \{y = Y\}]$$

Moreover we have $U > 0$ and $W > 0$ when $y > 0$, and $\partial U / \partial y > 0$, $\partial W / \partial y > 0$ when $y = 0$. The functions p_x / ρ and p_z / ρ are assumed to be known and negative.

T h e o r e m 1. Let the boundary value problem (1) have a solution $u, v, w \in C^2(\overline{G})$. Then $u \geqslant 0$ and $w \geqslant 0$ everywhere in $\overline{G}$ and the equalities $u = 0$ and $w = 0$ are attained only when $y = 0$.

P r o o f. Put $S(a) = \overline{G} \cap \{x = a\}$, $0 \leqslant a \leqslant X$. From the boundary conditions and the continuity of the functions u, w and their first derivatives it follows that a number $\varepsilon > 0$ exists such that when $y > 0$, then $u > 0$ and $w > 0$ for all $x \in [0; \varepsilon)$, $z \in [0; \varepsilon)$, $t \in [0; \varepsilon)$, $y \in (Y - \varepsilon; Y]$. Let $A = \sup a_1$ of the numbers a_1 such that $u > 0$ when $y > 0$ on all $S(a)$, $a \in [0; a_1)$. Clearly, $0 < A \leqslant X$. If $A < X$, then a sequence $a_n \to A$, $a_n > A$ exists such that at least one point $P_n \in S(a_n) \cap \{y > 0\}$ can be found for which $u(P_n) = 0$. Choosing for convenience a subsequence n_k, we can assume without restricting the generality, that $P_n \to P \in \overline{S(A)}$, and in this case we also have, by virtue of the continuity,

$u(P) = 0$. If $P^{(x)} = A$, $P^{(y)}$, $P^{(z)}$, $P^{(t)}$ are coordinates of the point P, then in view of the above arguments only the following cases are possible:

1) $0 < P^{(y)} \leqslant Y - \varepsilon$, $\varepsilon \leqslant P^{(z)} < Z$, $\varepsilon \leqslant P^{(t)} \leqslant T$. Then, since $u \geqslant 0$ in $\overline{S(A)}$, $u_{yy}(P) \geqslant 0$, $u_y(P) = u_z(P) = 0$, $u_t(P) \leqslant 0$ and the first equation of the system (1) cannot hold at the point P.

2) $P^{(y)} = 0$, $\varepsilon \leqslant P^{(z)} \leqslant Z$, $\varepsilon \leqslant P^{(t)} \leqslant T$. In this case $w(P) = 0$, $u_y(P) \geqslant 0$, $u_t(P) \leqslant 0$ and the first equation of the system (1) yields $\nu u_{yy}(P) = p_x/\rho + u_t(P) + V(P)u_y(P) < 0$. Now if $u_y(P) = 0$, then $u < 0$ on the interval $x = A$, $0 < y < y_0$, $z = P^{(z)}$, $t = P^{(t)}$ where y_0 is sufficiently small, and this contradicts the non-negativity of u on $S(A)$. If $u_y(P) > 0$, then by virtue of the continuity, $u_y > 0$ also on the set $\{A - \delta < x < A + \delta,\ y = 0,\ P^{(z)} - \delta < z < P^{(z)} + \delta,\ P^{(t)} - \delta < t < P^{(t)} + \delta\} \cap \overline{G}$ for some $\delta > 0$. From this it follows that $u > 0$ on the set $N = \{A - \delta < x < A + \delta, 0 < y < y_1, P^{(z)} - \delta < z < P^{(z)} + \delta,\ P^{(t)} - \delta < t < P^{(t)} + \delta\} \cap \overline{G}$ for some $y_1 > 0$. But at sufficiently large values of n all points P_n fall within the set N, and this again leads to a contradiction.

3) $0 < P^{(y)} < Y - \varepsilon$, $P^{(z)} = Z$, $\varepsilon \leqslant P^{(t)} \leqslant T$. Then $u_{yy}(P) \geqslant 0$, $u_y(P) = 0$, $u_z(P) \leqslant 0$, $u_t(P) \leqslant 0$ and we must assume, in order to avoid the contradiction with the first equation of the system (1), that $u_z(P) < 0$ and $w(P) < 0$. Let $B = \sup b_1$ of the numbers b_1 such that $w > 0$ when $y > 0$ for all $S(b)$, $b \in [0; b_1)$, and let $Q_n \to Q \in \overline{S(B)}$ be a sequence of points defined for w just as it was done for the function u. Putting $Q^{(y)} = 0$ leads to a contradiction as in Case 2. If, on the other hand, $0 < Q^{(y)} \leqslant Y - \varepsilon$, $\varepsilon \leqslant Q^{(z)} \leqslant Z$, $\varepsilon \leqslant Q^{(t)} \leqslant T$, then $w(Q) = 0$, $w_{yy}(Q) \geqslant 0$, $w_y(Q) = 0$, $w_x(Q) \leqslant 0$, $w_t(Q) \leqslant 0$, and since $B < A$, we have $u(Q) > 0$, which contradicts the second equation of (1) at the point Q.

Thus $A = X$ and similarly $B = X$. Therefore $u \geqslant 0$, $w \geqslant 0$ everywhere in $\bar{G}$ and the equality $u = 0$ for $y > 0$ can occur only at some point M such that $M^{(x)} = X$. But then $u_x(M) \leqslant 0$, $u_y(M) = 0$, $u_z(M) \leqslant 0$, $u_t(M) \leqslant 0$, $u_{yy}(M) \geqslant 0$ and we again reach a contradiction with the system (1). Simlilarly we show that $w > 0$ when $y > 0$, and this completes the proof of the theorem.

L e m m a. If u, v, w is a solution of class $C^2(\overline{G})$ of the boundary value problem (1), then any function $\varphi \in C^2(\overline{G})$ for which

$$L(\varphi) = \nu\varphi_{yy} - u\varphi_x - v\varphi_y - w\varphi_z - \varphi_t < 0$$

will not reach its minimum, neither inside G, nor at the part

$$\partial G \cap [\{x = X\} \cup \{z = Z\} \cup \{t = T\}] \cap \{0 < y < Y\}$$

of the boundary. Indeed, if φ attains its minimum as some point M within the region G or on the indicated part of the boundary, then $\varphi_{yy}(M) \geqslant 0$, $\varphi_x(M) \leqslant 0$, $\varphi_y(M) = 0$, $\varphi_z(M) \leqslant 0$, $\varphi_t(M) \leqslant 0$ and we arrive at a contradiction with the inequality $L(\varphi) < 0$ which proves the lemma.

T h e o r e m 2. Let u, w be a solution of class $C^2(\overline{G})$ of the boundary value problem (1). Then

$$0 \leqslant u \leqslant \max U + t \max(-p_x/\rho) \tag{2}$$

$$0 \leqslant w \leqslant \max W + t \max(-p_z/\rho) \tag{3}$$

P r o o f. Consider the function $\varphi = \max U + [\max(-p_x/\rho) + \varepsilon]t - u$ for an arbitrary $\varepsilon > 0$. On one hand we have

$$L(\varphi) = -\max(-p_x / \rho) - \varepsilon - p_x / \rho < 0,$$

i.e. φ satisfies the conditions of the lemma; on the other hand, we have $\varphi \geqslant 0$ at the part of the boundary at which the function φ should attain its minimum. Therefore $\varphi \geqslant 0$ everywhere in $\bar{G}$ and this proves (2), provided that we make subsequently ε tend to zero. The inequality (3) is proved in the same manner.

Theorem 3. Let u, w be a solution of class $C^2(\bar{G})$ of the boundary value problem (1). Then we have

$$\min\left\{\inf_{y>0} \frac{W}{U}; \min \frac{p_z}{p_x}\right\} \leqslant \frac{w}{u} \leqslant \max\left\{\sup_{y>0} \frac{W}{U}; \max \frac{p_z}{p_x}\right\}$$

everywhere in G.

Proof. Let us set $\alpha = \max\{\sup_{y>0} W/U; \max p_z/p_x\} + \varepsilon$ where $\varepsilon > 0$. Then for the function $\varphi = \alpha u - w$ we have $L(\varphi) = \alpha p_x/\rho - p_z/\rho < 0$, i.e. φ satisfies the conditions of the lemma. On the other hand, $\varphi \geqslant 0$ everywhere where φ can attain its minimum value. It follows that $\varphi \geqslant 0$ in the whole region G, and the right inequality is proved if we make ε tend to zero. We prove the left inequality in the same manner using the function

$$\varphi = w - (\min\{\inf_{y>0} W/U; \min p_z/p_x\} - \varepsilon)\, u$$

We note that Theorems 1 — 3 also hold when $Y = \infty$, provided that $u \to U$ and $w \to W$ as $y \to \infty$ uniformly in x, z and t.

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